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2018-03-01

Martikainen , H , Mourgoglou , M & Tolsa , X 2018 , ' Improved Cotlar's inequality in the context of local Tb theorems ' , Journal of Functional Analysis , vol. 274 , no. 5 , pp. 1255-1275 . https://doi.org/10.1016/j.jfa.2017.12.013

http://hdl.handle.net/10138/312551
https://doi.org/10.1016/j.jfa.2017.12.013

Journal of Functional Analysis 274 (2018) 1255–1275

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Improved Cotlar's inequality in the context of local Tb theorems

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A R T I C L E I N F O

Article history:
Received 20 January 2016
Accepted 28 December 2017
Available online 5 January 2018
Communicated by L. Saloff-Coste

MSC:
42B20

Keywords:
Non-homogeneous analysis
Local Tb theorems
Cotlar's inequality

A B S T R A C T

In the context of local Tb theorems with L^p testing conditions we prove an enhanced Cotlar's inequality. This is related to the problem of removing the so called buffer assumption of Hytönen–Nazarov, which is the final barrier for the full solution of S. Hofmann's problem. We also investigate the problem of extending the Hytönen–Nazarov result to non-homogeneous measures. We work not just with the Lebesgue measure but with measures μ in $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \le Cr^n$, $n \in (0,d]$. The range of exponents in the Cotlar type inequality depend on n. Without assuming buffer we get the full range of exponents $p,q \in (1,2]$ for measures with

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[1] H.M. was supported by the Academy of Finland through the grant 266262: Multiparameter dyadic harmonic analysis and probabilistic methods. He is a member of the Finnish Centre of Excellence in Analysis and Dynamics Research.

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[3] Research of M.M. was supported by the ERC grant 320501 of the European Research Council (FP7/2007–2013).
[4] X.T. was supported by the ERC grant 320501 of the European Research Council (FP7/2007–2013) and partially supported by MTM-2016-77635-P, MDM-2014-044 (MICINN, Spain), and by Marie Curie ITN MAnET (FP7-607647).

https://doi.org/10.1016/j.jfa.2017.12.013

$n \leq 1$, and in general we get $p, q \in [2 - \epsilon(n), 2]$, $\epsilon(n) > 0$. Consequences for (non-homogeneous) local Tb theorems are discussed.

1. Introduction

Let μ be a Radon measure on $\mathbb{R}^d$. We say that a function b_Q is an $L^p(\mu)$-admissible test function on a cube $Q \subset \mathbb{R}^d$ (with constant B_1), if

(1) $\operatorname{spt} b_Q \subset Q$,
(2) $\mu(Q) = \int_Q b_Q \, d\mu$, and
(3) $\int_Q |b_Q|^p \, d\mu \leq B_1 \mu(Q)$.

A long standing problem (even for the Lebesgue measure $\mu = dx$) asks whether the L^2 boundedness of a Calderón–Zygmund operator T follows if we are given $p, q \in (1, \infty)$, and for every cube Q an $L^p(\mu)$-admissible test function b_Q so that

$$\int_Q |Tb_Q|^{q'} \, d\mu \lesssim \mu(Q)$$

and an $L^q(\mu)$-admissible test function p_Q so that

$$\int_Q |T^* p_Q|^{p'} \, d\mu \lesssim \mu(Q).$$

In the case that both exponents are simultaneously small, i.e. $p, q < 2$ (or even $p < 2 = q$), this is still not known in this original form. However, Hytönen–Nazarov [6] showed in the Lebesgue measure case that the L^2 boundedness follows if one assumes the *buffered* testing conditions

$$\int_{2Q} |Tb_Q|^{q'} \, dx + \int_{2Q} |T^* p_Q|^{p'} \, dx \lesssim |Q|.$$

Notice that the estimate over $2Q$ is in fact equivalent to the same estimate over the whole space $\mathbb{R}^d$. A key thing in the Lebesgue measure case is that if $1/p + 1/q \leq 1$ (which includes the case $p = q = 2$), then the original testing conditions automatically imply the stronger buffered testing conditions by Hardy's inequality. The non-homogeneous version for $p = q = 2$ (without buffer) is by the first named author and Lacey [7].

The need for the buffer assumption is related to delicate problems in passing from maximal truncations to the original operator. In the Hytönen–Nazarov paper [6] the buffer is used in Lemma 3.2, which is a version of Cotlar's inequality in the local Tb setting (i.e. one needs to use the existence of the test functions to prove the Cotlar, not the boundedness of the operator which one does not know). In this paper we prove a more sophisticated Cotlar's inequality (Theorem 3.1), which works in the non-homogeneous setting and (for the first time) always allows some exponents $p, q < 2$. For measures satisfying $\mu(B(x,r)) \lesssim r$, the full range of exponents is obtained. This is our main result.

We also prove the related non-homogeneous local Tb theorem with these improved exponents, which is Theorem 4.6. Here we choose to use the new strategy via the big pieces Tb theorem and the good lambda method from the recent paper by the first two named authors and Vuorinen [8]. In the Calderón–Zygmund realm this technique currently requires antisymmetry.

The history of the various local Tb theorems (not covered above) is extremely vast including the original one by M. Christ [4] (with L^∞ assumptions), the non-homogeneous extension of this by Nazarov–Treil–Volberg [9] and the first one with L^p testing conditions for model operators by Auscher–Hofmann–Muscalu–Tao–Thiele [1]. We also mention Auscher–Yang [3], Auscher–Routin [2] and Hofmann [5]. For a more extensive survey of the developments we refer to [6] and [7] (see also [8]).

2. Notation and definitions

We say that a Radon measure μ on $\mathbb{R}^d$ is of degree $n \in (0,d]$ if for some constant $C_0 < \infty$ we have that $\mu(B(x,r)) \le C_0 r^n$ for all $x \in \mathbb{R}^d$ and $r > 0$.

We say that $K\colon \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\} \to \mathbb{C}$ is an n-dimensional Calderón–Zygmund kernel if for some $C < \infty$ and $\alpha \in (0,1]$ we have that

$$|K(x,y)| \le \frac{C}{|x-y|^n}, \qquad x \ne y,$$

$$|K(x,y) - K(x',y)| \le C\frac{|x-x'|^\alpha}{|x-y|^{n+\alpha}}, \qquad |x-y| \ge 2|x-x'|,$$

and

$$|K(x,y) - K(x,y')| \le C\frac{|y-y'|^\alpha}{|x-y|^{n+\alpha}}, \qquad |x-y| \ge 2|y-y'|.$$

Given a Radon measure ν in $\mathbb{R}^d$, possibly complex, we define

$$T\nu(x) = \int K(x,y)\, d\nu(y), \qquad x \in \mathbb{R}^d \setminus \operatorname{spt}\nu.$$

We also define $T\nu(x)$ as above for any $x \in \mathbb{R}^d$ whenever the integral on the right hand side makes sense. We say that T is an n-dimensional SIO (singular integral operator) with kernel K. Since the integral may not always be absolutely convergent for $x \in \operatorname{spt}\nu$, we consider the following ϵ-truncated operators T_ϵ, $\epsilon > 0$:

$$T_\epsilon \nu(x) = \int\limits_{|x-y|>\epsilon} K(x,y)\, d\nu(y), \qquad x \in \mathbb{R}^d.$$

The integral on the right hand side is absolutely convergent if, say, $|\nu|(\mathbb{R}^d) < \infty$.

For a positive Radon measure μ in $\mathbb{R}^d$ and $f \in L^1_{\mathrm{loc}}(\mu)$ we define

$$T_\mu f(x) = T(f\mu)(x), \qquad x \in \mathbb{R}^d \setminus \operatorname{spt}(f\mu),$$

and

$$T_{\mu,\epsilon} f(x) = T_\epsilon(f\mu)(x), \qquad x \in \mathbb{R}^d.$$

The integral defining $T_{\mu,\epsilon} f(x)$ is absolutely convergent if for example $f \in L^p(\mu)$ for some $1 \le p < \infty$ and μ is of degree n.

We say that T_μ is bounded in $L^p(\mu)$ if the operators $T_{\mu,\epsilon}$ are bounded in $L^p(\mu)$ uniformly in $\epsilon > 0$. Singular integral operators which are bounded in $L^2(\mu)$ are called Calderón–Zygmund operators (CZO). The boundedness of T_μ from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ is defined analogously.

Let $M(\mathbb{R}^d)$ denote the space of finite complex Radon measures in $\mathbb{R}^d$ equipped with the norm of total variation $\|\nu\| = |\nu|(\mathbb{R}^d)$. We say that T is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$ if there exists some constant $C < \infty$ so that for every $\nu \in M(\mathbb{R}^d)$ we have that

$$\sup_{\lambda>0} \lambda \cdot \mu(\{x \in \mathbb{R}^d \colon\ |T_\epsilon \nu(x)| > \lambda\}) \le C\|\nu\|$$

for all $\epsilon > 0$.

We still require the important concept of maximal truncations. If T is an SIO then the maximal operator T_* is defined by

$$T_*\nu(x) = \sup_{\epsilon>0} |T_\epsilon \nu(x)|, \qquad \nu \in M(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

and the δ-truncated maximal operators $T_{*,\delta}$ is

$$T_{*,\delta}\nu(x) = \sup_{\epsilon>\delta} |T_\epsilon \nu(x)|, \qquad \nu \in M(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

Like above, we also set

$$T_{\mu,*} f(x) = T_*(f\mu) \qquad \text{and} \qquad T_{\mu,*,\delta} f(x) = T_{*,\delta}(f\mu).$$

We need the following centred maximal functions with respect to balls and cubes:

$$M_\mu \nu(x) = \sup_{r>0} \frac{|\nu|(B(x,r))}{\mu(B(x,r))}, \qquad M_\mu(f) := M_\mu(f\mu),$$

and

$$M_\mu^{\mathcal{Q}} \nu(x) = \sup_{r>0} \frac{|\nu|(Q(x,r))}{\mu(Q(x,r))}, \qquad M_\mu^{\mathcal{Q}}(f) := M_\mu^{\mathcal{Q}}(f\mu).$$

The variant $M_{\mu,p} f := M_\mu(|f|^p)^{1/p}$ will also be used.

A cube $Q \subset \mathbb{R}^d$ is said μ-(a,b)-doubling (or just (a,b)-doubling if the measure μ is clear from the context) if

$$\mu(aQ) \le b\mu(Q),$$

where aQ is the cube concentric with Q with diameter $a\,\mathrm{diam}(Q)$. If μ is a measure of degree n, then for $b > a^n$ we have the following result about the existence of doubling cubes. For every $x \in \operatorname{spt}\mu$ and $c > 0$ there exist some (a,b)-doubling cube Q centred at x with $\ell(Q) \ge c$ (see Section 2.4 in [11]).

Given $t > 0$ we say that a cube $Q \subset \mathbb{R}^d$ has t-small boundary with respect to the measure μ if

$$\mu(\{x \in 5Q\colon\ \mathrm{dist}(x, \partial Q) \le \lambda \ell(Q)\}) \le t\lambda\mu(5Q)$$

for every $\lambda > 0$ (here $\ell(Q)$ is the side length of Q). The following Lemma (Lemma 9.43 in [11]) is important for us (notice that it holds for general Radon measures).

Lemma 2.1. *Let μ_1 and μ_2 be two Radon measures on $\mathbb{R}^d$. Let $t > 0$ be some constant big enough (depending only on d). Then, given a cube $Q \subset \mathbb{R}^d$, there exists a concentric cube Q' so that $Q \subset Q' \subset 1.1Q$ which has t-small boundary with respect to μ_1 and μ_2.*

The final notation used is as follows. We write $A \lesssim B$, if there is a constant $C > 0$ so that $A \le CB$. We may also write $A \sim B$ if $B \lesssim A \lesssim B$. For a set A we denote by $\mu\lfloor A$ the restriction of the measure μ to the set A. *All the appearing test functions are test functions with a uniform constant* B_1 (as in the beginning of the Introduction).

3. Cotlar's inequality

The following is our improved version of Cotlar's inequality in the context of local Tb theorems with L^p type testing conditions. Compare to the relatively simple Lemma 3.2 in [6] (this Lemma is the source of the buffer assumption in [6]). The corollaries related to the integrability properties of the maximal truncations $T_{\mu,*}b_Q$ are discussed after the proof.

Theorem 3.1. *Let μ be a measure of degree n on $\mathbb{R}^d$ and T be an n-dimensional SIO. Let b and t be large enough constants (depending only on d). Suppose $Q \subset \mathbb{R}^d$ is a fixed cube, $p, q \in (1, 2]$ and $\delta > 0$. We assume that there exists an $L^p(\mu)$-admissible test function b_Q in Q so that*

$$\int_Q |T_{\mu,\delta} b_Q|^{q'} \, d\mu \lesssim \mu(Q).$$

Furthermore, we assume that for every $(5, b)$-doubling cube $R \subset Q$ with t-small boundary there exists an $L^q(\mu)$-admissible test function p_R in R so that

$$\begin{cases} \int_R |T^*_{\mu,\delta} p_R|^{p'} \, d\mu \lesssim \mu(R), & \text{if } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{np}, \\ \int_{2R} |T^*_{\mu,\delta} p_R|^{p'} \, d\mu \lesssim \mu(R) & \text{otherwise}. \end{cases}$$

Then for every $\epsilon > \delta$ and $x \in (1 - \tau)Q$, $\tau > 0$, we have that

$$|T_{\mu,\epsilon} b_Q(x)| \lesssim_\tau M_\mu b_Q(x) + M^{\mathcal{Q}}_{\mu,p} b_Q(x) + M^{\mathcal{Q}}_{\mu,q'}(1_Q T_{\mu,\delta} b_Q)(x).$$

Proof. Fix $\tau > 0$ and $x \in (1 - \tau)Q$. Fix $\epsilon_0 > \delta$. Choose the smallest m such that the ball $B(x, 2^m \epsilon_0)$ is $(5C_d, b)$-doubling (where C_d is a large enough dimensional constant), and let $\epsilon = 2^m \epsilon_0$. A standard calculation shows that

$$|T_{\mu,\epsilon_0} b_Q(x) - T_{\mu,\epsilon} b_Q(x)| \lesssim M_\mu b_Q(x).$$

Therefore, it is enough to control $T_{\mu,\epsilon} b_Q(x)$. Suppose $\epsilon > C_d \ell(Q)$. Then we have that

$$T_{\mu,\epsilon} b_Q(x) = \int_{Q \cap B(x,\epsilon)^c} K(x, y) b_Q(y) \, d\mu(y) = \int_{\emptyset} K(x, y) b_Q(y) \, d\mu(y) = 0.$$

Suppose then that $c_\tau \ell(Q) \le \epsilon \le C_d \ell(Q)$. Then we have that

$$|T_{\mu,\epsilon} b_Q(x)| \lesssim_\tau \frac{1}{\ell(Q)^n} \int_{B(x, C_d \ell(Q))} |b_Q| \, d\mu \lesssim M_\mu b_Q(x).$$

Finally, assume that $\epsilon < c_\tau \ell(Q)$ for a small enough constant c_τ to be fixed. Define the Radon measure $\sigma_p = |b_Q|^p \, d\mu$. Choose a cube R centred at x so that it has t-small boundary with respect to μ and σ_p, and

$$B(x, \epsilon) \subset R \subset B(x, C_d \epsilon) \subset Q.$$

The last inclusion holds if c_τ is fixed small enough. Notice that

$$\mu(5R) \le \mu(B(x, 5C_d \epsilon)) \le b\mu(B(x, \epsilon)) \le b\mu(R).$$

Therefore, $R \subset Q$ is also a μ-$(5, b)$-doubling cube. This means that there exists a function p_R like in the assumptions.

For $z \in R$ we write

$$T_{\mu,\epsilon} b_Q(x) = T_{\mu,\epsilon} b_Q(x) - T_{\mu,\delta}(b_Q 1_{(2R)^c})(z) + T_{\mu,\delta} b_Q(z) - T_{\mu,\delta}(b_Q 1_{2R})(z).$$

For all $z \in R$ we have that

$$\begin{aligned}
&|T_{\mu,\epsilon} b_Q(x) - T_{\mu,\delta}(b_Q 1_{(2R)^c})(z)| \\
&\le \int\limits_{(2R)^c} |K(x,y) - K(z,y)|\, |b_Q(y)|\, d\mu(y) + \int\limits_{B(x,\epsilon)^c \cap (2R)} |K(x,y)|\, |b_Q(y)|\, d\mu(y) \\
&\lesssim \epsilon^\alpha \int\limits_{B(x,\epsilon)^c} \frac{|b_Q(y)|}{|x-y|^{n+\alpha}}\, d\mu(y) + \frac{1}{\epsilon^n} \int\limits_{B(x,2C_d\epsilon)} |b_Q(y)|\, d\mu(y) \lesssim M_\mu b_Q(x).
\end{aligned}$$

We now estimate

$$\begin{aligned}
|T_{\mu,\epsilon} b_Q(x)| &= \Big| \frac{1}{\mu(R)} \int\limits_R p_R(z) T_{\mu,\epsilon} b_Q(x)\, d\mu(z) \Big| \\
&\lesssim M_\mu b_Q(x) + \frac{1}{\mu(R)} \int\limits_R |p_R|\, |T_{\mu,\delta} b_Q|\, d\mu + \frac{1}{\mu(R)} \int\limits_{2R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu.
\end{aligned}$$

We have that

$$\begin{aligned}
\frac{1}{\mu(R)} \int\limits_R |p_R|\, |T_{\mu,\delta} b_Q|\, d\mu &\le \Big(\frac{1}{\mu(R)} \int\limits_R |p_R|^q\, d\mu \Big)^{1/q} \Big(\frac{1}{\mu(R)} \int\limits_R 1_Q |T_{\mu,\delta} b_Q|^{q'}\, d\mu \Big)^{1/q'} \\
&\lesssim M^{\mathcal{Q}}_{\mu,q'}(1_Q T_{\mu,\delta} b_Q)(x).
\end{aligned}$$

It remains to estimate

$$\frac{1}{\mu(R)} \int\limits_{2R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu.$$

Under the stronger assumption $\int_{2R} |T^*_{\mu,\delta} p_R|^{p'}\, d\mu \lesssim \mu(R)$ we can simply estimate as follows:

$$\begin{aligned}
\frac{1}{\mu(R)} \int\limits_{2R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu &\lesssim \Big(\frac{1}{\mu(R)} \int\limits_{2R} |T^*_{\mu,\delta} p_R|^{p'} \Big)^{1/p'} \Big(\frac{1}{\mu(2R)} \int\limits_{2R} |b_Q|^p\, d\mu \Big)^{1/p} \\
&\lesssim M^{\mathcal{Q}}_{\mu,p} b_Q(x).
\end{aligned}$$

In the previous argument we used that R is doubling.

Assume now only the weaker estimate $\int_R |T^*_{\mu,\delta} p_R|^{p'}\, d\mu \lesssim \mu(R)$. Then we write

$$\frac{1}{\mu(R)} \int\limits_{2R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu = \frac{1}{\mu(R)} \int\limits_{R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu + \frac{1}{\mu(R)} \int\limits_{2R\setminus R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu.$$

The first term is dominated by $M^{\mathcal{Q}}_{\mu,p} b_Q(x)$ using Hölder's inequality like above. The second term will be handled by a more tricky small boundaries trick (recall that R has t-small boundary with respect to the measure $\sigma_p = |b_Q|^p\, d\mu$). Denote also $\sigma = \sigma_1$ and $\nu_R = |p_R|\, d\mu$.

We begin by estimating

$$\begin{aligned}\int\limits_{2R\setminus R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu &= \int\limits_{2R\setminus R} \Bigg| \int\limits_{y:\ |y-z|>\delta} K(y,z) p_R(y)\, d\mu(y) \Bigg|\, d\sigma(z) \\ &\lesssim \int\limits_{R} \int\limits_{2R\setminus R} \frac{d\sigma(z)}{|z-y|^n}\, d\nu_R(y).\end{aligned}$$

Notice that $\mu(\partial R) = 0$ since R has t-small boundary with respect to μ. Fixing $y \in \operatorname{int} R$ we estimate

$$\begin{aligned}\int\limits_{2R\setminus R} \frac{d\sigma(z)}{|z-y|^n} &\le \sum_{j=0}^{\infty} \int\limits_{\{z\notin R:\ 2^{-j}\operatorname{diam}(R)\le|z-y|\le 2^{-j+1}\operatorname{diam}(R)\}} \frac{d\sigma(z)}{|z-y|^n} \\ &\lesssim \sum_{j=0}^{\infty} (2^{-j}\ell(R))^{-n} \sigma(B(y, 2^{-j+1}\operatorname{diam}(R)) \setminus R) \\ &= \sum_{j=0}^{\infty} \sum_{P\in\mathcal{D}_j(R)} 1_P(y)\ell(P)^{-n}\sigma(B(y, 2^{-j+1}\operatorname{diam}(R)) \setminus R) \\ &\le \sum_{\substack{P\in\mathcal{D}(R)\\ 5P\cap\partial R\neq\emptyset}} \frac{\sigma(5P)}{\ell(P)^n} 1_P(y).\end{aligned}$$

This yields

$$\begin{aligned}\int\limits_{2R\setminus R} |T^*_{\mu,\delta} p_R|\, |b_Q|\, d\mu &\lesssim \sum_{\substack{P\in\mathcal{D}(R)\\ 5P\cap\partial R\neq\emptyset}} \frac{\sigma(5P)\nu_R(P)}{\ell(P)^n} \\ &= \sum_{\substack{P\in\mathcal{D}(R)\\ 5P\cap\partial R\neq\emptyset}} \frac{\sigma(5P)}{\ell(P)^{n/2}} \eta_P^{-1/2} \cdot \frac{\nu_R(P)}{\ell(P)^{n/2}} \eta_P^{1/2}\end{aligned}$$

$$\lesssim \sum_{P\in\mathcal{D}(R)} \frac{\nu_R(P)^2}{\ell(P)^n}\eta_P + \sum_{\substack{P\in\mathcal{D}(R)\\ 5P\cap\partial R\neq\emptyset}} \frac{\sigma(5P)^2}{\ell(P)^n}\eta_P^{-1} = A + B.$$

Here we choose

$$\eta_P = \Big(\frac{\ell(P)}{\ell(R)}\Big)^u M^{\mathcal{Q}}_{\mu,p} b_Q(x)$$

for some yet to be fixed $u > 0$.

We begin by estimating the term A. We have

$$\begin{aligned}
A &= M^{\mathcal{Q}}_{\mu,p} b_Q(x) \sum_{P\in\mathcal{D}(R)} \frac{[\int_P |p_R|\,d\mu]^2}{\ell(P)^n}\Big(\frac{\ell(P)}{\ell(R)}\Big)^u \\
&\le M^{\mathcal{Q}}_{\mu,p} b_Q(x) \sum_{P\in\mathcal{D}(R)} \frac{[\mu(P)^{1/q'}(\int_P |p_R|^q\,d\mu)^{1/q}]^2}{\ell(P)^n}\Big(\frac{\ell(P)}{\ell(R)}\Big)^u \\
&\lesssim M^{\mathcal{Q}}_{\mu,p} b_Q(x) \Big[\int_R |p_R|^q\,d\mu\Big]^{2/q-1} \sum_{P\in\mathcal{D}(R)} \frac{\int_P |p_R|^q\,d\mu}{\ell(P)^{n(1-2/q')}}\Big(\frac{\ell(P)}{\ell(R)}\Big)^u \\
&\lesssim M^{\mathcal{Q}}_{\mu,p} b_Q(x) \frac{\mu(R)^{2/q-1}}{\ell(R)^{n(1-2/q')}} \int_R |p_R|^q\,d\mu \sum_{k=0}^{\infty} 2^{k(n-2n/q'-u)} \\
&\lesssim M^{\mathcal{Q}}_{\mu,p} b_Q(x)\mu(R),
\end{aligned}$$

provided that

$$u > n - \frac{2n}{q'}. \tag{3.2}$$

We then continue by estimating the term B. We have

$$\begin{aligned}
B &= \frac{1}{M^{\mathcal{Q}}_{\mu,p} b_Q(x)} \sum_{\substack{P\in\mathcal{D}(R)\\ 5P\cap\partial R\neq\emptyset}} \frac{[\int_{5P} |b_Q|\,d\mu]^2}{\ell(P)^n}\Big(\frac{\ell(R)}{\ell(P)}\Big)^u \\
&\lesssim \frac{1}{M^{\mathcal{Q}}_{\mu,p} b_Q(x)} \sum_{\substack{P\in\mathcal{D}(R)\\ 5P\cap\partial R\neq\emptyset}} \frac{\sigma_p(5P)^{2/p}}{\ell(P)^{n(1-2/p')}}\Big(\frac{\ell(R)}{\ell(P)}\Big)^u \\
&= \frac{1}{M^{\mathcal{Q}}_{\mu,p} b_Q(x)} \frac{1}{\ell(R)^{n(1-2/p')}} \sum_{k=0}^{\infty} 2^{ku} 2^{kn(1-2/p')} \sum_{\substack{P\in\mathcal{D}_k(R)\\ 5P\cap\partial R\neq\emptyset}} \sigma_p(5P)^{2/p}.
\end{aligned}$$

With a fixed k we estimate

$$\sum_{\substack{P\in\mathcal{D}_k(R)\\ 5P\cap\partial R\neq\emptyset}}\sigma_p(5P)^{2/p}\le\Big(\sum_{\substack{P\in\mathcal{D}_k(R)\\ 5P\cap\partial R\neq\emptyset}}\sigma_p(5P)\Big)^{2/p}=\Big(\int|b_Q|^p\Big[\sum_{\substack{P\in\mathcal{D}_k(R)\\ 5P\cap\partial R\neq\emptyset}}1_{5P}\Big]\,d\mu\Big)^{2/p}.$$

Notice then that here

$$5P\subset\{y\in 5R\colon\ d(y,\partial R)\le C2^{-k}\ell(R)\}$$

and

$$\sum_{P\in\mathcal{D}_k(R)}1_{5P}\lesssim 1.$$

Using that R has t-small boundary with respect to σ_p we can now deduce that

$$\sum_{\substack{P\in\mathcal{D}_k(R)\\ 5P\cap\partial R\neq\emptyset}}\sigma_p(5P)^{2/p}\lesssim\sigma_p(\{y\in 5R\colon\ d(y,\partial R)\le C2^{-k}\ell(R)\})^{2/p}\lesssim 2^{-2k/p}\sigma_p(5R)^{2/p}.$$

Noticing that

$$\sigma_p(5R)^{2/p}\le\mu(5R)^{2/p}M^{\mathcal{Q}}_{\mu,p}b_Q(x)^2\lesssim\ell(R)^{n(2/p-1)}\mu(R)M^{\mathcal{Q}}_{\mu,p}b_Q(x)^2$$

this yields that

$$B\lesssim M^{\mathcal{Q}}_{\mu,p}b_Q(x)\mu(R)\sum_{k=0}^{\infty}2^{k(-2/p+u+n-2n/p')}\lesssim M^{\mathcal{Q}}_{\mu,p}b_Q(x)\mu(R)$$

provided that

$$u<\frac{2}{p}+\frac{2n}{p'}-n. \tag{3.3}$$

Assuming that the constant u can be chosen appropriately we have proved that

$$\frac{1}{\mu(R)}\int_{2R\setminus R}|T^*_{\mu,\delta}p_R|\,|b_Q|\,d\mu\lesssim\frac{A+B}{\mu(R)}\lesssim M^{\mathcal{Q}}_{\mu,p}b_Q(x).$$

We see from (3.2) and (3.3) that the constant u can be chosen if

$$n-\frac{2n}{q'}<\frac{2}{p}+\frac{2n}{p'}-n.$$

This amounts precisely to

$$\frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{np}. \quad \square$$

The main implication is that the maximal truncation $T_{\mu,*}b_Q$ still satisfies reasonable testing conditions.

Corollary 3.4. *Let μ be a measure of degree n on $\mathbb{R}^d$ and T be an n-dimensional SIO. Let b and t be large enough constants (depending only on d). Suppose $Q \subset \mathbb{R}^d$ is a fixed cube, $p, q \in (1, 2]$ and $\delta > 0$. We assume that there exists an $L^p(\mu)$-admissible test function b_Q in Q so that*

$$\int_Q |T_{\mu,\delta} b_Q|^{q'} \, d\mu \lesssim \mu(Q).$$

Furthermore, we assume that for every $(5, b)$-doubling cube $R \subset Q$ with t-small boundary there exists an $L^q(\mu)$-admissible test function p_R in R so that

$$\begin{cases} \int_R |T^*_{\mu,\delta} p_R|^{p'} \, d\mu \lesssim \mu(R), & \text{if } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{np}, \\ \int_{2R} |T^*_{\mu,\delta} p_R|^{p'} \, d\mu \lesssim \mu(R) & \text{otherwise }. \end{cases}$$

Let $\tau > 0$ and $0 < a < p$. We have that

$$\int_{(1-\tau)Q} [T_{\mu,*,\delta} b_Q]^a \, d\mu \lesssim_{\tau,a} \mu(Q).$$

Proof. Using Theorem 3.1 we see that

$$\begin{aligned} \int_{(1-\tau)Q} [T_{\mu,*,\delta} b_Q]^a \, d\mu &\lesssim_\tau \int_Q [M_\mu b_Q]^a \, d\mu + \int_Q [M^{\mathcal{Q}}_{\mu,p} b_Q]^a \, d\mu + \int_Q [M^{\mathcal{Q}}_{\mu,q'}(1_Q T_{\mu,\delta} b_Q)]^a \, d\mu \\ &= I + II + III. \end{aligned}$$

Notice that $I \lesssim \mu(Q)$, which can be seen by using Hölder's inequality with the exponent $p/a > 1$ and the $L^p(\mu)$ boundedness of M_μ.

For the remaining terms II and III we shall use the inequality

$$\int_Q |f|^a \, d\mu \le \frac{s}{s-a} \mu(Q)^{1-a/s} \|f\|^a_{L^{s,\infty}(\mu)}, \qquad s > a. \tag{3.5}$$

Using (3.5) with $s = p > a$ we see that

$$II \lesssim_a \mu(Q)^{1-a/p} \|M^{\mathcal{Q}}_{\mu,p} b_Q\|^a_{L^{p,\infty}(\mu)} = \mu(Q)^{1-a/p} \|M^{\mathcal{Q}}_{\mu}(|b_Q|^p)\|^{a/p}_{L^{1,\infty}(\mu)}$$
$$\lesssim \mu(Q)^{1-a/p} \|b_Q\|^a_{L^p(\mu)} \lesssim \mu(Q),$$

where we used that $M^{\mathcal{Q}}_{\mu}$ maps $L^1(\mu) \to L^{1,\infty}(\mu)$ boundedly.

Similarly, using (3.5) with $s = q' \geq 2 > a$ we see that

$$III \lesssim_a \mu(Q)^{1-a/q'} \|M^{\mathcal{Q}}_{\mu}(1_Q |T_{\mu,\delta} b_Q|^{q'})\|^{a/q'}_{L^{1,\infty}(\mu)} \lesssim \mu(Q)^{1-a/q'} \|1_Q T_{\mu,\delta} b_Q\|^a_{L^{q'}(\mu)} \lesssim \mu(Q).$$

This ends the proof. $\square$

In the following corollary we record the fully symmetric statement.

Corollary 3.6. *Let μ be a measure of degree n on $\mathbb{R}^d$ and T be an n-dimensional SIO. Let b and t be large enough constants (depending only on d), and $p, q \in (1, 2]$. For every $(5, b)$-doubling cube $Q \subset \mathbb{R}^d$ with t-small boundary we assume that there exist an $L^p(\mu)$-admissible test function b_Q in Q so that*

$$\begin{cases} \sup_{\delta>0} \int_Q |T_{\mu,\delta} b_Q|^{q'} \, d\mu \lesssim \mu(Q), & \textit{if } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{nq}, \\ \sup_{\delta>0} \int_{2Q} |T_{\mu,\delta} b_Q|^{q'} \, d\mu \lesssim \mu(Q) & \textit{otherwise}\ , \end{cases}$$

and an $L^q(\mu)$-admissible test function p_Q in Q so that

$$\begin{cases} \sup_{\delta>0} \int_Q |T^*_{\mu,\delta} p_Q|^{p'} \, d\mu \lesssim \mu(Q), & \textit{if } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{np}, \\ \sup_{\delta>0} \int_{2Q} |T^*_{\mu,\delta} p_Q|^{p'} \, d\mu \lesssim \mu(Q) & \textit{otherwise}\ . \end{cases}$$

Let $\tau > 0$. Then for every $(5, b)$-doubling cube $Q \subset \mathbb{R}^d$ with t-small boundary we have

$$\int_{(1-\tau)Q} [T_{\mu,*} b_Q]^a \, d\mu \lesssim_{\tau,a} \mu(Q), \qquad 0 < a < p,$$

and

$$\int_{(1-\tau)Q} [T^*_{\mu,*} p_Q]^a \, d\mu \lesssim_{\tau,a} \mu(Q), \qquad 0 < a < q.$$

Remark 3.7. Notice that if $n = 1$ the condition

$$1/p + 1/q < 1 + 1/(np) = 1 + 1/p$$

only says that $q > 1$ (and the symmetric condition only says that $p > 1$) yielding the full range of exponents without buffer. In general, one can have both $p, q < 2$ simultaneously without buffer, showcasing that $p = q = 2$ is not a threshold after which one is required to assume buffer.

4. Implications to local Tb theorems

It is easier to prove local Tb theorems assuming conditions for maximal truncations $T_{\mu,*}b_Q$ rather than $T_\mu b_Q$. In fact, there is a tradeoff here. One needs much weaker conditions on $T_{\mu,*}b_Q$ compared to $T_\mu b_Q$, but of course $T_{\mu,*}b_Q$ is a larger object to begin with. Probably most convenient is to prove a local Tb theorem assuming conditions on $T_{\mu,*}b_Q$, and then reduce the one what with operator testing to this via Corollary 3.6. The point of the maximal truncations is to allow suppression arguments. In our proof these suppression arguments are hidden to the big pieces Tb theorem (originally by Nazarov–Treil–Volberg [10]) that we apply. The method of proof in [6] also involves suppression (in a different way) and the proof is not directly applicable in the non-homogeneous situation.

We want to adapt the convenient strategy from the recent paper by the first two named authors and Vuorinen [8]. This new strategy via the big pieces Tb theorem and non-homogeneous good lambda method is ideal in the square function setting, since there is no duality and no maximal truncations in that context. Because extending the big pieces Tb theorem to concern all Calderón–Zygmund operators seems difficult (it only currently works for antisymmetric ones), we make the antisymmetry assumption here.

4.1. Big pieces via maximal truncations

The next Proposition (Proposition 4.2) with testing assumptions about maximal truncations corresponds to Proposition 2.3 in [8]. The proof from that setting can be directly moved here, and as such one could make the assumptions as weak as in [8]. For the convenience of the reader we quickly reprove a less general statement here (if one is interested in as general a statement as possible, just look at [8]). This will be enough for deriving the local Tb theorem with operator testing, which is our main focus here.

Definition 4.1. Given a cube $Q \subset \mathbb{R}^d$ we consider the following random dyadic grid. For small notational convenience assume that $c_Q = 0$ (that is, Q is centred at the origin). Let $N \in \mathbb{Z}$ be defined by the requirement $2^{N-3} \le \ell(Q) < 2^{N-2}$. Consider the random square $Q^* = Q^*(w) = w + [-2^N, 2^N)^n$, where $w \in [-2^{N-1}, 2^{N-1})^d =: \Omega_N = \Omega$. The set Ω is equipped with the normalised Lebesgue measure $\mathbb{P}_N = \mathbb{P}$. We define the grid $\mathcal{D}(w) := \mathcal{D}(Q^*(w))$ (the local dyadic grid generated by the cube $Q^*(w)$). Notice that $Q \subset \alpha Q^*(w)$ for some $\alpha < 1$, and $\ell(Q) \sim \ell(Q^*(w))$.

Proposition 4.2. *Let μ be a measure of degree n on $\mathbb{R}^d$ and T be an n-dimensional SIO with a kernel K satisfying $K(x, y) = -K(y, x)$. Let $Q \subset \mathbb{R}^d$ be a fixed cube, $q \in (1, \infty)$*

and b_Q be an $L^q(\mu)$-admissible test function in Q with constant B_1. Then there exists a small constant $c_1 = c_1(q, B_1) > 0$ with the following property. If there exist $s > 0$ and an exceptional set $E_Q \subset \mathbb{R}^d$ so that $\int_{E_Q} |b_Q|\, d\mu \le c_1 \int_Q |b_Q|\, d\mu$ and

$$\sup_{\lambda>0} \lambda^s \mu(\{x \in Q \setminus E_Q \colon T_{\mu,*}b_Q(x) > \lambda\}) \le B_2\mu(Q) \text{ for some } B_2 < \infty, \tag{4.3}$$

then there exists $G_Q \subset Q \setminus E_Q$ so that $\mu(G_Q) \gtrsim \mu(Q)$ and $T_{\mu\lfloor G_Q} \colon L^2(\mu\lfloor G_Q) \to L^2(\mu\lfloor G_Q)$ with a norm depending on the constants in the assumptions.

Proof. We can assume that $\operatorname{spt}\mu \subset Q$. Indeed, if we have proved the theorem for such measures, we can then apply it to $\mu\lfloor Q$. Let us define the measure σ by setting $\sigma(A) = \int_A |b_Q|\, d\mu$. Also, write $b_Q = |b_Q|\widehat{b}_Q$ using the polar decomposition, so that $|\widehat{b}_Q| = 1$. The big pieces Tb theorem by Nazarov–Treil–Volberg (Theorem 5.1 in [11]) will be applied to the measure σ and the L^∞-function $\widehat{b}_Q$. (Notice that Theorem 5.1 in [11] is stated for the Cauchy operator, but holds true for all antisymmetric CZO with the same proof. Moreover, the L^1 testing assumption there can directly be weakened to a weak type testing condition.)

We fix w, and write $\mathcal{D}(w) = \mathcal{D}$. We also write $\mathcal{D}_0 = \mathcal{D}(0)$. Let $\mathcal{A} = \mathcal{A}_w$ consist of the maximal dyadic cubes $R \in \mathcal{D}$ for which

$$\Big| \int_R \widehat{b}_Q \, d\sigma \Big| < \eta\sigma(R),$$

where $\eta := \frac{1}{2}B_1^{-1/q}$. We set

$$T = T_w = \bigcup_{R\in\mathcal{A}} R \subset \mathbb{R}^d.$$

Notice that

$$\sigma(Q) = \int_Q |b_Q|\, d\mu \le B_1^{1/q}\mu(Q) = B_1^{1/q} \int_Q b_Q \, d\mu = B_1^{1/q} \int_Q \widehat{b}_Q \, d\sigma.$$

Then estimate

$$\int_Q \widehat{b}_Q \, d\sigma = \int_{Q\setminus T} \widehat{b}_Q \, d\sigma + \sum_{R\in\mathcal{A}} \int_R \widehat{b}_Q \, d\sigma \le \sigma(Q \setminus T) + \eta\sigma(Q).$$

Since $\eta B_1^{1/q} = 1/2$ we conclude that

$$\sigma(Q) \le B_1^{1/q}\sigma(Q \setminus T) + \frac{1}{2}\sigma(Q),$$

and so

$$\sigma(Q) \le 2B_1^{1/q}[\sigma(Q) - \sigma(T)].$$

From here we can read that

$$\sigma(T) \le \tau_0 \sigma(Q), \qquad \tau_0 := 1 - \frac{1}{2B_1^{1/q}} = 1 - \eta < 1.$$

Next, let $\mathcal{F}$ consist of the maximal dyadic cubes $R \in \mathcal{D}_0$ for which

$$\int_R |b_Q|^q \, d\mu > C_0 \mu(R)$$

or

$$\sigma(R) < \delta \mu(R),$$

where $C_0 := [16B_1\eta^{-1}]^{q'}$ and $\delta := \eta/16$. Let $\mathcal{F}_1$ be the collection of maximal cubes $R \in \mathcal{D}_0$ satisfying the first condition, and define $\mathcal{F}_2$ analogously. Note that

$$\mu\Big(\bigcup_{R \in \mathcal{F}_1} R\Big) \le B_1 C_0^{-1} \mu(Q),$$

and so

$$\begin{aligned}
\sigma\Big(\bigcup_{R \in \mathcal{F}_1} R\Big) &= \int_{\bigcup_{R \in \mathcal{F}_1} R} |b_Q| \, d\mu \\
&\le \mu\Big(\bigcup_{R \in \mathcal{F}_1} R\Big)^{1/q'} \Big(\int_Q |b_Q|^q \, d\mu\Big)^{1/q} \\
&\le [B_1 C_0^{-1}]^{1/q'} \mu(Q)^{1/q'} \cdot B_1^{1/q} \mu(Q)^{1/q} = \delta\mu(Q) \le \delta\sigma(Q).
\end{aligned}$$

Finally, we record that

$$\sigma\Big(\bigcup_{R \in \mathcal{F}_2} R\Big) = \sum_{R \in \mathcal{F}_2} \sigma(R) \le \delta \sum_{R \in \mathcal{F}_2} \mu(R) = \delta\mu\Big(\bigcup_{R \in \mathcal{F}_2} R\Big) \le \delta\mu(Q) \le \delta\sigma(Q).$$

We may conclude that the set

$$H_1 = \bigcup_{R \in \mathcal{F}} R$$

satisfies $\sigma(H_1) \le 2\delta\sigma(Q) = \frac{\eta}{8}\sigma(Q)$.

We now record the important property of the exceptional set H_1. Let $x \in Q \setminus H_1$. For any $R \in \mathcal{D}_0$ satisfying that $x \in R$ we have that

$$\delta \le \frac{\sigma(R)}{\mu(R)} = \frac{1}{\mu(R)} \int_R |b_Q| \, d\mu \le \Big(\frac{1}{\mu(R)} \int_R |b_Q|^q \, d\mu \Big)^{1/q} \le C_0^{1/q}.$$

Letting $\ell(R) \to 0$ we conclude that for μ-a.e. $x \in Q \setminus H_1$ we have $|b_Q(x)| \sim 1$.

We need another exceptional set H_2. To this end, let

$$p(x) = \sup_{r>0} \frac{\sigma(B(x,r))}{r^n} = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |b_Q| \, d\mu = M_\mu^R b_Q(x).$$

For $p_0 > 0$ let $E_{p_0} = \{p \ge p_0\}$. Notice that

$$\mu(E_{p_0}) = \mu(\{M_\mu^R b_Q \ge p_0\}) \le \frac{1}{p_0^q} \int [M_\mu^R b_Q]^q \, d\mu \lesssim \frac{1}{p_0^q} \mu(Q),$$

and so

$$\sigma(E_{p_0}) \le \mu(E_{p_0})^{1/q'} \Big(\int_Q |b_Q|^q \, d\mu \Big)^{1/q} \lesssim \frac{1}{p_0^{q-1}} \mu(Q) \le \frac{1}{p_0^{q-1}} \sigma(Q).$$

We fix p_0 so large that $\sigma(E_{p_0/2^n}) \le \frac{\eta}{8}\sigma(Q)$. For $x \in \{p > p_0\}$ define

$$r(x) = \sup\{r > 0 \colon \sigma(B(x,r)) > p_0 r^n\},$$

and then set

$$H_2 := \bigcup_{x \in \{p > p_0\}} B(x, r(x)).$$

It is clear that every ball B_r with $\sigma(B_r) > p_0 r^n$ satisfies $B_r \subset H_2$. Notice that if $y \in H_2$, then there is $x \in \{p > p_0\}$ so that $y \in B(x, r(x))$, and so $\sigma(B(y, 2r(x)) \ge \sigma(B(x, r(x)) \ge p_0 r(x)^n = p_0 2^{-n} [2r(x)]^n$. We conclude that $H_2 \subset E_{p_0/2^n}$, and so $\sigma(H_2) \le \frac{\eta}{8}\sigma(Q)$.

We can take $c_1 = \eta/8$ on the statement of the theorem. This means that $\sigma(E_Q) \le \frac{\eta}{8}\sigma(Q)$. Define now $H = H_1 \cup H_2 \cup E_Q$. The properties of H are as follows:

(1) We have $\sigma(H) \le \frac{\eta}{2}\sigma(Q)$, and so $\sigma(H \cup T_w) \le \big(1 - \frac{\eta}{2}\big)\sigma(Q) = \tau_1 \sigma(Q)$, $\tau_1 < 1$.
(2) If $\sigma(B_r) > p_0 r^n$, then $B_r \subset H$.
(3) $|b_Q(x)| \sim 1$ for μ-a.e. $x \in Q \setminus H$.

We also have for every $\lambda > 0$ that

$$\lambda^s \sigma(\{x \in Q \setminus H \colon T_{\sigma,*} \widehat{b}_Q(x) > \lambda\})$$

$$= \lambda^s \sigma(\{x \in Q \setminus H \colon\ T_{\mu,*}b_Q(x) > \lambda\})$$
$$= \lambda^s \int\limits_{\{x \in Q\setminus H \colon\ T_{\mu,*}b_Q(x)>\lambda\}} |b_Q|\, d\mu$$
$$\lesssim \lambda^s \mu(\{x \in Q \setminus E_Q \colon\ T_{\mu,*}b_Q(x) > \lambda\}) \le B_2 \mu(Q) \lesssim \sigma(Q).$$

Appealing to the big pieces global Tb theorem by Nazarov–Treil–Volberg (Theorem 5.1 in [11]) with the measure σ and the *bounded* function $\widehat{b}_Q$ we find $G_Q \subset Q \setminus H \subset Q \setminus E_Q$ so that $\sigma(G_Q) \gtrsim \sigma(Q)$ and

$$\sup_{\epsilon>0} \|1_{G_Q} T_{\sigma,\epsilon} f\|_{L^2(\sigma)} \lesssim \|f\|_{L^2(\sigma)} \tag{4.4}$$

for every $f \in L^2(\sigma)$ satisfying $\operatorname{spt} f \subset G_Q$.

Let $\epsilon > 0$. Suppose now that $g \in L^2(\mu)$ and $\operatorname{spt} g \subset G_Q$. We apply Equation (4.4) with $f = g/|b_Q|$ (since $G_Q \subset Q \setminus H$ we have $|b_Q| \sim 1$ on the support of g). Notice that

$$\|1_{G_Q} T_{\sigma,\epsilon}(g/|b_Q|)\|_{L^2(\sigma)} = \|1_{G_Q} T_{\mu,\epsilon} g\|_{L^2(\sigma)} \gtrsim \|1_{G_Q} T_{\mu,\epsilon} g\|_{L^2(\mu)}$$

so that

$$\|1_{G_Q} T_{\mu,\epsilon} g\|_{L^2(\mu)} \lesssim \|g/|b_Q|\|_{L^2(\sigma)} \lesssim \|g\|_{L^2(\mu)}.$$

Since $\epsilon > 0$ was arbitrary this means precisely that $T_{\mu\lfloor G_Q} \colon L^2(\mu\lfloor G_Q) \to L^2(\mu\lfloor G_Q)$ boundedly. Moreover, we have that

$$\mu(Q) \le \sigma(Q) \lesssim \sigma(G_Q) = \int\limits_{G_Q} |b_Q|\, d\mu \lesssim \mu(G_Q).$$

We are done. $\square$

We record as a corollary a local Tb theorem with maximal truncations testing. Again, this could be improved as in [8], but our main focus is the local Tb theorem on the next subsection (only the previous proposition is needed for that).

Corollary 4.5. *Let μ be a measure of degree n on $\mathbb{R}^d$ and T be an n-dimensional SIO with a kernel K satisfying $K(x, y) = -K(y, x)$. Suppose $q \in (1, \infty)$, and let b and t be large enough constants (depending only on d). We assume that to every $(5, b)$-doubling cube $Q \subset \mathbb{R}^d$ with t-small boundary there is associated an $L^q(\mu)$-admissible test function b_Q in Q with constant B_1 such that*

$$\sup_{\lambda>0} \lambda^s \mu(\{x \in Q \colon\ T_{\mu,*}b_Q(x) > \lambda\}) \le B_2 \mu(Q) \text{ for some } B_2 < \infty \text{ and } s > 0.$$

Then $T_\mu \colon L^2(\mu) \to L^2(\mu)$ with a bound depending on the above constants.

Proof. Fix an arbitrary $(5, b)$-doubling cube $Q \subset \mathbb{R}^d$ with t-small boundary. By the good lambda method (Theorem A.1 and Remark A.2) it is enough to show that there exists $G_Q \subset Q$ so that $\mu(G_Q) \gtrsim \mu(Q)$ and $T_{\mu\lfloor G_Q} \colon L^2(\mu\lfloor G_Q) \to L^2(\mu\lfloor G_Q)$. By Proposition 4.2 this follows from the assumptions. □

4.2. Local Tb theorem with operator testing

Theorem 4.6. *Let μ be a measure of degree n on $\mathbb{R}^d$ and T be an n-dimensional SIO with a kernel K satisfying $K(x, y) = -K(y, x)$. Suppose $q \in (1, 2]$, and let b and t be large enough constants (depending only on d). We assume that to every $(5, b)$-doubling cube $Q \subset \mathbb{R}^d$ with t-small boundary there is associated an $L^q(\mu)$-admissible test function b_Q in Q with constant B_1 such that*

$$\begin{cases} \sup_{\delta>0} \int_Q |T_{\mu,\delta} b_Q|^{q'} \, d\mu \lesssim \mu(Q), & \text{if } \frac{1}{q} < \frac{1}{2}\left(1 + \frac{1}{nq}\right), \\ \sup_{\delta>0} \int_{2Q} |T_{\mu,\delta} b_Q|^{q'} \, d\mu \lesssim \mu(Q) & \text{otherwise .} \end{cases}$$

Then $T_\mu \colon L^2(\mu) \to L^2(\mu)$ with a bound depending on the above constants.

Proof. Fix an arbitrary $(5, b)$-doubling cube $Q \subset \mathbb{R}^d$ with t-small boundary. By the good lambda method (Theorem A.1 and Remark A.2) it is enough to show that there exists $G_Q \subset Q$ so that $\mu(G_Q) \gtrsim \mu(Q)$ and $T_{\mu\lfloor G_Q} \colon L^2(\mu\lfloor G_Q) \to L^2(\mu\lfloor G_Q)$. By Proposition 4.2 it is enough to show that

$$\int_{Q\setminus E_Q} T_{\mu,*} b_Q \, d\mu \lesssim \mu(Q)$$

for some set $E_Q \subset \mathbb{R}^d$ satisfying that $\int_{E_Q} |b_Q| \, d\mu \le c_1 \int_Q |b_Q| \, d\mu$, where $c_1 = c_1(B_1, q) > 0$.

Let $E_Q = Q \setminus (1 - \tau_0)Q$ for some $\tau_0 < 1$ large enough. Then we have, since Q has t-small boundary and is doubling, that

$$\mu(E_Q) \le \epsilon(\tau_0)\mu(Q),$$

where $\lim_{\tau_0 \to 1} \epsilon(\tau_0) = 0$. In particular, we have that

$$\begin{aligned} \int_{E_Q} |b_Q| \, d\mu \le \mu(E_Q)^{1/q'} \|b_Q\|_{L^q(\mu)} &\le \epsilon(\tau_0)^{1/q'} B_1^{1/q} \mu(Q) \\ &= \epsilon(\tau_0)^{1/q'} B_1^{1/q} \int_Q b_Q \, d\mu \end{aligned}$$

$$\leq \epsilon(\tau_0)^{1/q'} B_1^{1/q} \int_Q |b_Q| \, d\mu \leq c_1 \int_Q |b_Q| \, d\mu$$

provided that $\tau_0 = \tau_0(B_1, q) < 1$ is fixed close enough to 1.

Now the estimate

$$\int_{Q \setminus E_Q} T_{\mu,*} b_Q \, d\mu = \int_{(1-\tau_0)Q} T_{\mu,*} b_Q \, d\mu \lesssim \mu(Q)$$

follows from Corollary 3.6, and we are done. □

Appendix A. Good lambda method with small boundaries

We prove a version of Theorem 2.22 from [11], which is weaker in the sense that we require only cubes with small boundaries.

Theorem A.1. *Let μ be a Radon measure on $\mathbb{R}^d$ of degree n and T be an n-dimensional SIO. Let $b > 0$ and C_1 be big enough (depending only on d) and let $\theta > 0$. Suppose that for every $(5, b)$-doubling cube Q with C_1-small boundary there exists some subset $G_Q \subset Q$, with $\mu(G_Q) \geq \theta \, \mu(Q)$, such that T_* is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu\lfloor G_Q)$, with norm bounded uniformly on Q. Then T_μ is bounded in $L^p(\mu)$, for $1 < p < \infty$, with its norm depending on p and on the preceding constants.*

Remark A.2. One can also assume that $T_{\mu\lfloor G_Q} \colon L^2(\mu\lfloor G_Q) \to L^2(\mu\lfloor G_Q)$ with norm bounded uniformly on Q, since then T_* is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu\lfloor G_Q)$ by standard results (see e.g. Theorem 2.21 in [11]).

To prove Theorem A.1 we will use a Whitney's decomposition of some open set. In the next lemma we show the precise version of the required decomposition.

Lemma A.3. *If $\Omega \subset \mathbb{R}^d$ is open, $\Omega \neq \mathbb{R}^d$, then Ω can be decomposed as*

$$\Omega = \bigcup_{i \in I} Q_i,$$

where Q_i, $i \in I$, are closed dyadic cubes with disjoint interiors such that for some constants $R > 20$ and $D_0 \geq 1$ depending only on d the following holds:

(i) *$10Q_i \subset \Omega$ for each $i \in I$.*
(ii) *$RQ_i \cap \Omega^c \neq \varnothing$ for each $i \in I$.*
(iii) *For each cube Q_i, there are at most D_0 cubes Q_j such that $10Q_i \cap 10Q_j \neq \varnothing$. Further, for such cubes Q_i, Q_j, we have $\ell(Q_i) \approx \ell(Q_j)$.*

Moreover, if μ is a positive Radon measure on $\mathbb{R}^d$ and $\mu(\Omega) < \infty$, there is a family of cubes $\{\widetilde{Q}_j\}_{j\in S}$, with $S \subset I$, so that $Q_j \subset \widetilde{Q}_j \subset 1.1Q_j$, satisfying the following:

(a) *Each cube $\widetilde{Q}_j$, $j \in S$, is $(9, 2D_0)$-doubling and has C_1-small boundary.*
(b) *The cubes $\widetilde{Q}_j$, $j \in S$, are pairwise disjoint.*
(c)

$$\mu\Big(\bigcup_{j\in S} \widetilde{Q}_j\Big) \geq \frac{1}{8D_0}\,\mu(\Omega). \tag{A.4}$$

Proof. Whitney's decomposition into dyadic cubes satisfying (i), (ii) and (iii) is a well known result.

To prove the existence of the family of $\{\widetilde{Q}_j\}_{j\in S}$, we denote by $I_{db} \subset I$ the subfamily of the indices such that the cubes from $\{Q_i\}_{i\in I_{db}}$ are $(10, 2D_0)$-doubling. Then notice that

$$\mu(Q_j) < \frac{1}{2D_0}\mu(10Q_j) \qquad \text{if } j \in I \setminus I_{db}.$$

Since

$$\sum_{j\in I} 1_{10Q_j} \leq D_0 1_\Omega,$$

we deduce that

$$\sum_{j\in I\setminus I_{db}} \mu(Q_j) \leq \frac{1}{2D_0}\sum_{j\in I}\mu(10Q_j) \leq \frac{1}{2}\mu(\Omega).$$

Thus,

$$\mu\Big(\bigcup_{j\in I_{db}} Q_j\Big) \geq \mu(\Omega) - \sum_{j\in I\setminus I_{db}} \mu(Q_j) \geq \frac{1}{2}\,\mu(\Omega),$$

and we can choose a finite subcollection $I^1_{db} \subset I_{db}$ so that

$$\mu\Big(\bigcup_{j\in I^1_{db}} Q_j\Big) \geq \frac{1}{4}\,\mu(\Omega). \tag{A.5}$$

By the covering lemma with triple cubes (see e.g. Theorem 2.1 in [11]), there exists a subfamily $S \subset I^1_{db}$ such that the cubes $\{2Q_j\}_{j\in S}$ are pairwise disjoint, and

$$\bigcup_{j\in I^1_{db}} Q_j \subset \bigcup_{j\in I^1_{db}} 2Q_j \subset \bigcup_{j\in S} 6Q_j.$$

For each $j \in S$, we consider a cube $\widetilde{Q}_j$ with $Q_j \subset \widetilde{Q}_j \subset 1.1Q_j$ with a C_1-small boundary. Such a cube exists e.g. by Lemma 9.43 in [11].

Clearly, the cubes $\widetilde{Q}_j$, $j \in S$, are pairwise disjoint by construction. Further,

$$\mu(9\widetilde{Q}_j) \leq \mu(10Q_j) \leq 2D_0\,\mu(Q_j) \leq 2D_0\,\mu(\widetilde{Q}_j).$$

This means that the cubes are $(9, 2D_0)$-doubling as claimed. The proof of (c) is also easy, using (A.5) and the doubling property of the cubes $\{Q_j\}_{j\in S}$:

$$\begin{aligned}\mu(\Omega) &\leq 4\,\mu\Big(\bigcup_{j\in I^1_{db}} Q_j\Big) \leq 4\,\mu\Big(\bigcup_{j\in S} 6Q_j\Big)\\ &\leq 4\sum_{j\in S}\mu(6Q_j) \leq 8D_0\sum_{j\in S}\mu(Q_j) \leq 8D_0\sum_{j\in S}\mu(\widetilde{Q}_j) = 8D_0\mu\Big(\bigcup_{j\in S}\widetilde{Q}_j\Big). \quad \square\end{aligned}$$

Proof of Theorem A.1. To prove the theorem we just have to adapt the arguments in Theorem 2.22 from [11] with very minor changes. Indeed, almost all changes reduce to replacing the cubes Q_i, $i \in S$, in the proof of Theorem 2.22 from [11] by the cubes $\widetilde{Q}_i$, $i \in S$, from Lemma A.3 (with $\Omega \equiv \Omega_\lambda$), and to replace the sum $\sum_{i\in I\setminus S}\mu(Q_i)$ appearing in various places of that proof by

$$\mu\Big(\Omega_\lambda \setminus \bigcup_{i\in S}\widetilde{Q}_i\Big).$$

The details are omitted. $\square$

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